

ON INFINITESIMAL DEFORMATIONS OF CMC SURFACES OF FINITE TYPE IN THE 3-SPHERE

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ABSTRACT. We describe infinitesimal deformations of constant mean curvature surfaces of finite type in the 3-sphere. We use Baker-Akhiezer functions to describe such deformations, as well as polynomial Killing fields and the corresponding spectral curve to distinguish between isospectral and non-isospectral deformations.

INTRODUCTION

The theory of constant mean curvature (CMC) surfaces, and more generally that of harmonic maps has developed greatly over the past decades. One reason for this may be that there are two main approaches possible towards the subject which cross fertilize each other: geometric PDE methods and integrable system techniques. In the late 1990's the integrable system approach culminated in a very general (local) description of such harmonic maps in terms of loop Lie algebra valued 1-forms by Dorfmeister, Pedit and Wu. In particular, in the case of CMC tori, Pinkall and Sterling [9], and independently Hitchin [8] showed that a solution to the structure equation (the sinh-Gordon equation) can be represented by a hyperelliptic Riemann surface, the so called spectral curve. The solution in this case is said to be of finite type.

Pinkall and Sterling [9] construct infinitesimal deformations of CMC tori in \mathbb{R}^3 . We carry over their constructions to CMC tori in \mathbb{S}^3 . Such tori in \mathbb{S}^3 have non-isospectral deformations changing the mean curvature in contrast to CMC tori in \mathbb{R}^3 . For this reason we consider also deformations that change the mean curvature. The corresponding normal variation then obeys an inhomogeneous Jacobi equation.

We briefly outline the contents of the paper: In the first section we recall some facts about CMC surfaces in \mathbb{S}^3 and introduce our notation. In particular, we recall the notion of extended frame and spectral curve for CMC surfaces of finite type. In the second section we construct Jacobi fields and parametric Jacobi fields for CMC tori in \mathbb{S}^3 . In the third section we construct homogenous Jacobi fields in terms of the Baker-Akhiezer function of the sinh-Gordon equation. Finally we show that the Fermi curve of the Jacobi operator is isomorphic to the spectral curve of the sinh-Gordon equation. In the last section we extend the deformation to a deformation of the corresponding polynomial Killing field, and exhibit both isospectral, and non-isospectral deformations in terms of polynomial Killing fields.

1. CONFORMAL CMC IMMERSIONS INTO \mathbb{S}^3

1.1. Extended frames. We identify the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with $\mathbb{S}^3 \cong (\mathrm{SU}(2) \times \mathrm{SU}(2)) / \mathrm{D}$, where D is the diagonal in $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The Lie algebra of the matrix Lie group $\mathrm{SU}(2)$ is $\mathfrak{su}(2)$, equipped with the commutator $[\cdot, \cdot]$. For $\alpha, \beta \in \Omega^1(T\mathbb{R}^2, \mathfrak{su}(2))$ smooth 1-forms on $\mathbb{R}^2 \cong T\mathbb{R}^2$ with values in $\mathfrak{su}(2)$, we define the $\mathfrak{su}(2)$ -valued 2-form

$$[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)],$$

for $X, Y \in T\mathbb{R}^2$. Let $L_g : h \mapsto gh$ be left multiplication in $\mathrm{SU}(2)$. Then by left translation, the tangent bundle is $T\mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathfrak{su}(2)$ and $\theta : T\mathrm{SU}(2) \rightarrow \mathfrak{su}(2)$, $v_g \mapsto (dL_{g^{-1}})_g v_g$ is the (left) Maurer–Cartan form. It satisfies the Maurer–Cartan equation

$$2d\theta + [\theta \wedge \theta] = 0. \quad (1.1)$$

For a map $F : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$, the pullback $\alpha = F^*\theta$ also satisfies (1.1), and conversely, every solution $\alpha \in \Omega^1(\mathbb{R}^2, \mathfrak{su}(2))$ of (1.1) integrates to a smooth map $F : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$ with $\alpha = F^*\theta$.

Complexifying the tangent bundle $T\mathbb{R}^2$ and decomposing into $(1, 0)$ and $(0, 1)$ tangent spaces, and writing $d = \partial + \bar{\partial}$, we may split $\omega \in \Omega^1(M, \mathfrak{su}(2))$ into the $(1, 0)$ part ω' , the $(0, 1)$ part ω'' and write $\omega = \omega' + \omega''$. We set the $*$ -operator on $\Omega^1(M, \mathfrak{sl}(2, \mathbb{C}))$ to $*\omega = -i\omega' + i\omega''$. Fix $\epsilon \in \mathfrak{su}(2)$ and let $\mathbb{T} = \mathrm{stab}(\epsilon)$ be the stabilizer of ϵ under the adjoint action of $\mathrm{SU}(2)$ on $\mathfrak{su}(2)$. We shall view the 2-sphere \mathbb{S}^2 as the quotient $\mathbb{S}^2 \cong \mathrm{SU}(2)/\mathbb{T}$.

We denote by $\langle \cdot, \cdot \rangle$ the bilinear extension of the Ad-invariant inner product of $\mathfrak{su}(2)$ to $\mathfrak{su}(2)^\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$ such that $\langle \epsilon, \epsilon \rangle = 1$. The double cover of the isometry group $\mathrm{SO}(4)$ is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ via the action $X \mapsto FXG^{-1}$. Writing $df = df' + df''$ for the differential of f , recall that an immersion $f : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$ is conformal if and only if

$$\langle df', df' \rangle = 0. \quad (1.2)$$

If $f : \mathbb{R}^2 \rightarrow \mathbb{S}^3$ is a conformal immersion and $\omega = f^{-1}df$, then it can be shown (see e.g. [10]) that the mean curvature function H of f is given by

$$2d*\omega = H[\omega \wedge \omega]. \quad (1.3)$$

Suppose $f : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$ is a conformal immersion with non-zero constant mean curvature H . Then $2d*\omega = H[\omega \wedge \omega]$ and $2d\omega + [\omega \wedge \omega] = 0$ combined give $d*\omega + H^{-1}d*\omega = 0$, or alternatively

$$(1 - iH^{-1})d\omega' + (1 + iH^{-1})d\omega'' = 0. \quad (1.4)$$

Inserting $2d\omega'' = -2d\omega' - [\omega \wedge \omega]$ respectively $2d\omega' = -2d\omega'' - [\omega \wedge \omega]$ into (1.4) gives $4d\omega' = (iH - 1)[\omega \wedge \omega]$ and $4d\omega'' = -(1 + iH)[\omega \wedge \omega]$. Then

$$\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})(1 + iH)\omega' + \frac{1}{2}(1 - \lambda)(1 - iH)\omega''$$

satisfies $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ for all $\lambda \in \mathbb{C}^*$, and thus there exists a corresponding *extended frame* $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \rightarrow \mathrm{SU}(2)$ with $dF_\lambda = F_\lambda \alpha_\lambda$ and $F_\lambda(0) = 1$. Conversely, we recall the following version of a result by Bobenko [2]. Our formulas are slightly different to those of Bobenko [1, 2, 3], but the modifications in the proof are obvious.

Theorem 1.1. *Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function and define*

$$\alpha_\lambda = \frac{1}{2} \begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1}e^u dz + i e^{-u} d\bar{z} \\ i e^{-u} dz + i \lambda e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}. \quad (1.5)$$

Then $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ if and only if u is a solution of the sinh-Gordon equation

$$\partial\bar{\partial}u + \frac{1}{2}\sinh(2u) = 0. \quad (1.6)$$

Furthermore, for any solution u of the sinh-Gordon equation and corresponding extended frame $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \rightarrow \mathrm{SU}(2)$, and $\lambda_0, \lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$, the map $f : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$ defined by

$$f = F_{\lambda_1} F_{\lambda_0}^{-1} \quad (1.7)$$

is a conformal immersion with constant mean curvature

$$H = i \frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}, \quad (1.8)$$

conformal factor

$$v^2 = \frac{e^{2u}}{H^2 + 1}, \quad (1.9)$$

and Hopf differential $Q dz^2$ with

$$Q = \frac{i}{4} (\lambda_1^{-1} - \lambda_0^{-1}). \quad (1.10)$$

Corollary 1.2. *Useful formulae obtained from equations (1.8), (1.9) and (1.10) are*

$$v^2 = 4e^{2u}Q\bar{Q}, \quad 4Q\bar{Q}(H^2 + 1) = 1. \quad (1.11)$$

The constant mean curvature surface f in \mathbb{S}^3 corresponding to a solution u of the sinh-Gordon equation satisfies the following equations with respect to the moving frame $(f, \partial f, \bar{\partial} f, N)$:

$$\begin{aligned} \partial N &= -H \partial f + 2v^{-2}Q \bar{\partial} f \\ \partial \partial f &= 2u_z \partial f - Q N \\ 2\partial \bar{\partial} f &= -v^2 f + v^2 H N \end{aligned} \quad (1.12)$$

1.2. Spectral curve. Let $f : \mathbb{R}^2 \rightarrow \mathrm{SU}(2)$ be a conformal CMC immersion with corresponding extended frame F_λ . For a translation $\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we write $\tau^* f = f \circ \tau$, and define the monodromy of F_λ with respect to τ as

$$M(\tau, \lambda) = \tau^* F_\lambda F_\lambda^{-1}.$$

Now suppose we have a conformal CMC immersion of a torus $f : \mathbb{R}^2/\Gamma \rightarrow \mathrm{SU}(2)$, with lattice

$$\Gamma = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z},$$

and corresponding extended frame F_λ . If $M_1(\lambda), M_2(\lambda)$ are the monodromies of F_λ with respect to ω_1 and ω_2 , and μ_1, μ_2 the corresponding eigenvalues, then the *spectral curve* of the torus is

$$\Sigma_f = \{(\lambda, \mu_1, \mu_2) : \det(\mu_1 \mathbb{1} - M_1(\lambda)) = \det(\mu_2 \mathbb{1} - M_2(\lambda)) = 0\}.$$

We recall the description of CMC tori in terms of spectral curves:

Let Y be a hyperelliptic Riemann surface with branch points over $\lambda = 0(y^+)$ and $\lambda = \infty(y^-)$. Then Y is the spectral curve of an immersed CMC torus in the three sphere if and only if the following four conditions hold:

- (i) Besides the hyperelliptic involution σ , the surface Y has two further anti-holomorphic involutions η and $\varrho = \eta \circ \sigma = \sigma \circ \eta$, such that η has no fixed points and $\eta(y^+) = y^-$.
- (ii) There exist two non-zero holomorphic functions μ_1, μ_2 on $Y \setminus \{y^+, y^-\}$ such that for $i = 1, 2$

$$\sigma^* \mu_i = \mu_i^{-1}, \eta^* \bar{\mu}_i = \mu_i, \varrho^* \bar{\mu}_i = \mu_i^{-1}.$$

- (iii) The forms $d \ln \mu_i$ are meromorphic differentials of the second kind with double poles at y^\pm . The singular parts at y^+ respectively y^- of these two differentials are linearly independent.
- (iv) There are four fix points $y_1, y_2 = \sigma(y_1), y_3, y_4 = \sigma(y_3)$ of ϱ , such that the functions μ_1 and μ_2 are either 1 or -1 there.

We shall describe spectral curves of CMC tori in \mathbb{S}^3 via hyperelliptic surfaces of the form

$$\nu^2 = \begin{cases} \lambda a(\lambda) & \text{if } g \text{ is even,} \\ a(\lambda) & \text{if } g \text{ is odd.} \end{cases}$$

Here $\lambda : Y \rightarrow \mathbb{CP}^1$ is chosen so that y^\pm correspond to $\lambda = 0$ respectively $\lambda = \infty$, and

$$\sigma^* \lambda = \lambda, \eta^* \bar{\lambda} = \lambda^{-1}, \varrho^* \bar{\lambda} = \lambda^{-1},$$

and

$$a(\lambda) = \prod_{i=1}^g (\lambda - \alpha_i)(\lambda^{-1} - \bar{\alpha}_i)$$

with pair-wise different branch points $\alpha_1, \dots, \alpha_g \in \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Hence $\eta^* \bar{a} = a$ and $\varrho^* \bar{a} = a$. For $|\lambda| = 1$ we have that $a(\lambda) > 0$, and since η has no fix points, we have

$$\begin{cases} \eta^* \bar{\nu} = -\lambda^{-1} \nu, & \varrho^* \bar{\nu} = \lambda^{-1} \nu, & \sigma^* \nu = -\nu & \text{if } g \text{ is even,} \\ \eta^* \bar{\nu} = -\nu, & \varrho^* \bar{\nu} = \nu, & \sigma^* \nu = -\nu & \text{if } g \text{ is odd.} \end{cases}$$

Up to now, the parameter λ is only determined up to a rotation. Pick $\lambda_0, \lambda_1 \in \mathbb{S}^1$, $\lambda_1 \neq 1$, and take the unique parameter λ for which the points y_1 and $y_2 = \sigma(y_1)$ correspond to the two points over $\lambda = \lambda_0$. Then the points y_3 and $y_4 = \sigma(y_3)$ correspond to the two points over $\lambda = \lambda_1$.

2. JACOBI FIELDS

An infinitesimal deformation of a CMC surface f by CMC surfaces is given by a normal vector field $\dot{f} = \omega N$, where the function $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ has to be a solution of a Jacobi equation. For such a deformation in Euclidean 3-space, it turns out that ω and the infinitesimal change of the conformal factor solve the same homogeneous Jacobi equation. We shall see that this is not the case for CMC surfaces in the 3-sphere. The reason for this is that the mean curvature changes throughout a deformation in a non-trivial way, introducing an inhomogeneity in the Jacobi equation for ω .

Assume we are given a smooth one-parameter family $f_t : \mathbb{R}^2 \rightarrow \text{SU}(2)$, $t \in (-\epsilon, \epsilon)$ of CMC surfaces of finite type, with appropriate solutions u_t of the sinh-Gordon equation, mean

curvatures H_t , and Hopf differentials $Q_t dz^2$, and normals N_t . Denoting differentiation with respect to t by a dot, and omitting the subscript t when $t = 0$, we then have

$$\dot{f} = \tau \partial f + \sigma \bar{\partial} f + \omega N \quad (2.1)$$

with smooth functions $\tau, \sigma : \mathbb{C} \rightarrow \mathbb{C}$ such that $\bar{\tau} = \sigma$, and a smooth function $\omega : \mathbb{C} \rightarrow \mathbb{R}$.

The following is the analogue of Proposition 2.1 in Pinkall and Sterling [9].

Proposition 2.1. *Every Jacobi field ωN along f can be supplemented by a tangential component $\tau \partial f + \sigma \bar{\partial} f$ to yield a parametric Jacobi field. Further, if (2.1) is a parametric Jacobi field, then ω solves the inhomogeneous Jacobi equation*

$$\bar{\partial} \partial \omega + \cosh(2u) \omega = \frac{\dot{H} e^{2u}}{2(H^2 + 1)}, \quad (2.2)$$

while \dot{u} solves the homogeneous Jacobi equation

$$\bar{\partial} \partial \dot{u} + \cosh(2u) \dot{u} = 0. \quad (2.3)$$

Proof. Straightforward computations using (1.12) give

$$\begin{aligned} \partial \dot{f} &= -\frac{\sigma v^2}{2} f + (\partial \tau + 2\tau \partial u - \omega H) \partial f + \left(\partial \sigma + \frac{2\omega Q}{v^2}\right) \bar{\partial} f + \left(\frac{\sigma v^2}{2} H - \tau Q + \partial \omega\right) N, \\ \bar{\partial} \dot{f} &= -\frac{\tau v^2}{2} f + (\bar{\partial} \tau + \frac{2\omega \bar{Q}}{v^2}) \partial f + (\bar{\partial} \sigma + 2\sigma \bar{\partial} u - \omega H) \bar{\partial} f + \left(\frac{\tau v^2}{2} H - \sigma \bar{Q} + \bar{\partial} \omega\right) N. \end{aligned}$$

Conformality $\langle \partial f, \partial f \rangle = \langle \bar{\partial} f, \bar{\partial} f \rangle = 0$ implies that $\langle \partial \dot{f}, \partial f \rangle = \langle \bar{\partial} \dot{f}, \bar{\partial} f \rangle = 0$, and consequently for τ, σ to supplement ωN to a parametric Jacobi field they must satisfy

$$\begin{aligned} \partial \sigma &= -2v^{-2} \omega Q, \\ \bar{\partial} \tau &= -2v^{-2} \omega \bar{Q}. \end{aligned} \quad (2.4)$$

Hence the above expressions for $\partial \dot{f}$ and $\bar{\partial} \dot{f}$ simplify to

$$\begin{aligned} \partial \dot{f} &= -\frac{\sigma v^2}{2} f + (\partial \tau + 2\tau \partial u - \omega H) \partial f + \left(\frac{\sigma v^2}{2} H - \tau Q + \partial \omega\right) N, \\ \bar{\partial} \dot{f} &= -\frac{\tau v^2}{2} f + (\bar{\partial} \tau + 2\sigma \bar{\partial} u - \omega H) \bar{\partial} f + \left(\frac{\tau v^2}{2} H - \sigma \bar{Q} + \bar{\partial} \omega\right) N. \end{aligned} \quad (2.5)$$

Differentiating $v^2 = 2\langle \partial f, \bar{\partial} f \rangle$ gives

$$v \dot{v} = \langle \partial \dot{f}, \bar{\partial} f \rangle + \langle \partial f, \bar{\partial} \dot{f} \rangle = \frac{1}{2} v^2 (\partial \tau + 2\tau \partial u + \bar{\partial} \sigma + 2\sigma \bar{\partial} u - 2\omega H),$$

and combining this with the derivative of equation (1.9) yields

$$\dot{u} = \frac{1}{2} \partial \tau + \tau \partial u + \frac{1}{2} \bar{\partial} \sigma + \sigma \bar{\partial} u - \omega H + \frac{H \dot{H}}{H^2 + 1}. \quad (2.6)$$

From (1.12) we have $2\langle \bar{\partial} \partial f, N \rangle = v^2 H$, and differentiating this gives on the one hand

$$\langle \bar{\partial} \partial \dot{f}, N \rangle + \langle \bar{\partial} \partial f, \dot{N} \rangle = v \dot{v} H + \frac{v^2}{2} \dot{H}.$$

Differentiating $2\bar{\partial} \partial f = -v^2 f + v^2 H N$ gives

$$2\bar{\partial} \partial \dot{f} = -2v \dot{v} f - v^2 \dot{f} + 2v \dot{v} H N + v^2 \dot{H} N + v^2 H \dot{N}.$$

Hence $2\langle \bar{\partial}\partial\dot{f}, N \rangle = -v^2\omega + 2v\dot{v}H + v^2\dot{H}$, and consequently

$$2\langle \bar{\partial}\partial f, \dot{N} \rangle = v^2\omega.$$

On the other hand, differentiating the first equation of (2.5) with respect to \bar{z} , and then computing $\langle \bar{\partial}\partial\dot{f}, N \rangle$ gives

$$\begin{aligned} \langle \bar{\partial}\partial\dot{f}, N \rangle &= \frac{1}{2}(\partial\tau + 2\tau\partial u - \omega H)v^2H - \bar{Q}(\partial\sigma + 2\omega v^{-2}Q) \\ &\quad + \frac{1}{2}v^2H\bar{\partial}\sigma + \sigma v\bar{\partial}vH - \bar{\partial}\tau Q + \bar{\partial}\partial\omega, \end{aligned}$$

and equating this with the first expression we obtained for $2\langle \bar{\partial}\partial\dot{f}, N \rangle$, we obtain (2.2).

Equation (2.3) is now proven by a direct calculation using equations (2.6), (2.2) and the time derivative of the second equation in (1.11).

It remains to prove that the differential equations defining σ, τ are integrable. From

$$\begin{aligned} \langle \dot{N}, f \rangle &= -\langle N, \dot{f} \rangle = -\omega, \\ \langle \dot{N}, \partial f \rangle &= -\langle N, \partial\dot{f} \rangle = \tau Q - \sigma \frac{v^2}{2}H - \partial\omega, \\ \langle \dot{N}, \bar{\partial}f \rangle &= -\langle N, \bar{\partial}\dot{f} \rangle = \sigma Q - \tau \frac{v^2}{2}H - \bar{\partial}\omega, \end{aligned}$$

we compute

$$\dot{N} = -\omega f + v^{-2}(2\sigma Q - \tau v^2H - 2\bar{\partial}\omega)\partial f + v^{-2}(2\tau Q - \sigma v^2H - 2\partial\omega)\bar{\partial}f. \quad (2.7)$$

Differentiating $Q = -\langle \partial^2 f, N \rangle$ we have $\dot{Q} = -\langle \partial^2\dot{f}, N \rangle - \langle \partial^2 f, \dot{N} \rangle$, and with (2.7) learn that

$$\langle \partial^2 f, \dot{N} \rangle = (2\tau Q - \sigma v^2H - 2\partial\omega)\partial u.$$

Differentiating the first equation of (2.5) with respect to z and taking inner product against N gives

$$\langle \partial^2\dot{f}, N \rangle = \omega QH - 2Q\partial\tau - 2Q\tau\partial u + \frac{1}{2}v^2H\partial\sigma + \sigma vH\partial v + \partial^2\omega,$$

and therefore $\dot{Q} = 2Q\partial\tau + 2\partial u\partial\omega - \partial^2\omega$. Solving this equation for $\partial\tau$ gives

$$\partial\tau = \frac{\dot{Q} + \partial^2\omega - 2\partial u\partial\omega}{2Q}. \quad (2.8)$$

Finally, a straightforward computation proves that integrability for τ holds automatically, that is differentiating $\bar{\partial}\tau$ in (2.4) with respect to z gives the same as differentiating $\partial\tau$ in (2.8) with respect to \bar{z} . Also, the analogous statement then holds for σ , that is differentiating

$$\bar{\partial}\sigma = \frac{\dot{\bar{Q}} + \bar{\partial}^2\omega - 2\bar{\partial}u\bar{\partial}\omega}{2\bar{Q}}. \quad (2.9)$$

with respect to z gives the same as differentiating $\partial\sigma$ in (2.4) with respect to \bar{z} . Hence the functions τ, σ that supplement the Jacobi field are unique up to addition of complex constants. We compute τ, σ explicitly in terms of Baker-Akhiezer functions in (3.3). \square

The following is the analogue of Proposition 2.2 in Pinkall and Sterling [9]. We call a parametric Jacobi field a Killing field, if it is induced by an infinitesimal isometry of \mathbb{S}^3 .

Proposition 2.2. *A parametric Jacobi field is a Killing field if and only if $\dot{u} = 0$.*

Proof. The 'only if' part is trivial. A parametric Jacobi field (2.1) is a Killing field, if and only if there exists elements $g_0, g_1 \in \mathfrak{su}(2)$, such that

$$\dot{f} = g_1 f - f g_0 = F_{\lambda_1} (F_{\lambda_1}^{-1} g_1 F_{\lambda_1} - F_{\lambda_0}^{-1} g_0 F_{\lambda_0}) F_{\lambda_0}^{-1}. \quad (2.10)$$

Setting $B_0 = F_{\lambda_0}^{-1} g_0 F_{\lambda_0}$, $B_1 = F_{\lambda_1}^{-1} g_1 F_{\lambda_1}$, then equation (2.10) reads

$$F_{\lambda_1}^{-1} \dot{f} F_{\lambda_0} = B_1 - B_0 = \tau (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) + \sigma (\alpha''_{\lambda_1} - \alpha''_{\lambda_0}) + \omega \epsilon.$$

The derivatives together with equations (2.5) yield

$$\begin{aligned} F_{\lambda_1}^{-1} \partial \dot{f} F_{\lambda_0} &= B_1 (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) - (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) B_0 \\ &= -\frac{1}{2} \sigma v^2 \mathbb{1} + (\partial \tau + 2\tau \partial u - \omega H) (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) + (\frac{1}{2} \sigma v^2 H - \tau Q + \partial \omega) \epsilon, \\ F_{\lambda_1}^{-1} \bar{\partial} \dot{f} F_{\lambda_0} &= B_1 (\alpha''_{\lambda_1} - \alpha''_{\lambda_0}) - (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) B_0 \\ &= -\frac{1}{2} \sigma v^2 \mathbb{1} + (\bar{\partial} \sigma + 2\sigma \bar{\partial} u - \omega H) (\alpha''_{\lambda_1} - \alpha''_{\lambda_0}) + (\frac{1}{2} \tau v^2 H - \sigma \bar{Q} + \bar{\partial} \omega) \epsilon. \end{aligned}$$

These equations can be solved and we obtain

$$\begin{aligned} B_0 &= \frac{1}{2i} (\partial \tau + 2\tau \partial u - \omega(i + H)) \epsilon + v^{-2} (i\sigma \bar{Q} - \frac{1}{2} \tau v^2 (1 + iH) - i\bar{\partial} \omega) (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) \\ &\quad - v^{-2} (i\tau Q + \frac{1}{2} \sigma v^2 (1 - iH) - i\partial \omega) (\alpha''_{\lambda_1} - \alpha''_{\lambda_0}), \end{aligned}$$

$$\begin{aligned} B_1 &= \frac{1}{2i} (\partial \tau + 2\tau \partial u + \omega(i - H)) \epsilon + v^{-2} (i\sigma \bar{Q} + \frac{1}{2} \tau v^2 (1 - iH) - i\bar{\partial} \omega) (\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) \\ &\quad - v^{-2} (i\tau Q - \frac{1}{2} \sigma v^2 (1 + iH) - i\partial \omega) (\alpha''_{\lambda_1} - \alpha''_{\lambda_0}). \end{aligned}$$

Consequently, setting $g_j = F_{\lambda_j} B_j F_{\lambda_j}^{-1}$ for $j = 0, 1$, we get

$$\begin{aligned} f g_0 &= \frac{1}{2i} (\partial \tau + 2\tau \partial u - \omega(i + H)) N + v^{-2} (i\sigma \bar{Q} - \frac{1}{2} \tau v^2 (1 + iH) - i\bar{\partial} \omega) \partial f \\ &\quad - v^{-2} (i\tau Q + \frac{1}{2} \sigma v^2 (1 - iH) - i\partial \omega) \bar{\partial} f, \\ g_1 f &= \frac{1}{2i} (\partial \tau + 2\tau \partial u + \omega(i - H)) N + v^{-2} (i\sigma \bar{Q} + \frac{1}{2} \tau v^2 (1 - iH) - i\bar{\partial} \omega) \partial f \\ &\quad - v^{-2} (i\tau Q - \frac{1}{2} \sigma v^2 (1 + iH) - i\partial \omega) \bar{\partial} f. \end{aligned}$$

It remains to prove that $\dot{u} = 0$ implies that $dg_1 = dg_2 = 0$, or equivalently that both $d(fg_0) = df f^{-1} f g_0$ and $d(g_1 f) = g_1 f f^{-1} df$ hold. To this end it is useful to recall the identities

$$\begin{aligned} \partial f f^{-1} \partial f &= 0, & \bar{\partial} f f^{-1} \bar{\partial} f &= 0, \\ \bar{\partial} f f^{-1} \partial f &= -\frac{1}{2} v^2 (f + iN), & \partial f f^{-1} \bar{\partial} f &= -\frac{1}{2} v^2 (f - iN), \\ N f^{-1} \partial f &= -\partial f f^{-1} N = i \partial f, & -N f^{-1} \bar{\partial} f &= \bar{\partial} f f^{-1} N = i \bar{\partial} f. \end{aligned}$$

The rest of the proof is now a straightforward computation. \square

3. THE HOMOGENEOUS JACOBI FIELDS

We next calculate a solution of the homogeneous Jacobi equation (2.2). Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a solution of the sinh-Gordon equation. Let $\psi = (\psi_1, \psi_2)^t$ be a solution of

$$\begin{aligned} \partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} \partial u & ie^{-u} \\ i\lambda^{-1}e^u & -\partial u \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} -\bar{\partial} u & i\lambda e^u \\ ie^{-u} & \bar{\partial} u \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned} \quad (3.1)$$

Then $\varphi = (\varphi_1, \varphi_2)^t$ defined by

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

solves

$$\begin{aligned} \partial \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} \partial u & i\lambda^{-1}e^u \\ ie^{-u} & -\partial u \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \\ \bar{\partial} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} -\bar{\partial} u & i\lambda e^u \\ i\lambda e^u & \bar{\partial} u \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \end{aligned} \quad (3.2)$$

For every value of $\lambda \in \mathbb{C}^*$ there exists a two dimensional space of such functions ψ and φ . Assume u is periodic with period γ . Then we assume in addition to (3.1) and (3.2) that these functions are eigenvalues of the translations by the period γ . Hence we get for any pair (λ, μ) , such that $\mu \neq \pm 1$ is an eigenvalue of the monodromy of ψ , a one dimensional space space of such functions ψ . A normalization of this function extends to a unique function ψ on the spectral curve called Baker-Akhiezer function. Since the monodromy of ψ has determinant one, there always exist another solution $\sigma^*\psi$ with inverse eigenvalue of the monodromy. We shall assume that the translations by the periods acts on φ with inverse eigenvalues as on ψ . This of course can be only achieved with φ obtained from another solution $\sigma^*\psi$ of (3.1). So we may assume that both $\psi_1\varphi_1$ and $\psi_2\varphi_2$ have trivial multipliers with respect to γ .

Proposition 3.1. *Let $\psi = (\psi_1, \psi_2)^t$ be a solution of (3.1) and let $\varphi = (\varphi_1, \varphi_2)^t$ be a solution of (3.2). Define $\omega = \psi_1\varphi_1 - \psi_2\varphi_2$ and $\phi = \partial\omega - 2Q\tau$. Then*

(i) *The function $y = \psi^t\varphi$ satisfies $dy = 0$.*

(ii) *The function ω is in the kernel of the Jacobi operator, and can be supplemented to a parametric Jacobi field with corresponding (up to complex constants)*

$$\tau = \frac{i\psi_2\varphi_1}{e^u Q}, \quad \sigma = -\frac{i\psi_1\varphi_2}{e^u \bar{Q}}. \quad (3.3)$$

(iii) *For the parametric Jacobi field generated by ω , τ and σ , the variation of the conformal factor is*

$$\dot{u} = \frac{(\lambda - \lambda_0)(\lambda - \lambda_1)}{\lambda(\lambda_0 - \lambda_1)} i\omega. \quad (3.4)$$

(iv) *Then $(\partial\omega)^2 - \phi^2 = \lambda^{-1}(y^2 - \omega^2)$.*

(v) *Further, $\partial\phi = 2\partial u\partial\omega$, and $\lambda^{-1}\omega = 2\phi\partial u - \partial^2\omega$.*

Proof. (i) is a straightforward computation using (3.1) and (3.2). To prove (ii), we compute

$$\begin{aligned}\partial\omega &= ie^{-u}\psi_2\varphi_1 - i\lambda^{-1}e^u\psi_1\varphi_2, \\ \bar{\partial}\partial\omega &= -\omega \cosh(2u), \\ \bar{\partial}(ie^{-u}\psi_2\varphi_1 + i\lambda^{-1}e^u\psi_1\varphi_2) &= \omega \sinh(2u).\end{aligned}\tag{3.5}$$

Hence ω is in the kernel of the Jacobi operator. From the $\bar{\partial}$ -derivative (2.4) of τ and

$$\begin{aligned}\bar{\partial}(-2Q\tau + \partial\omega) &= -2Q\bar{\partial}\tau + \bar{\partial}\partial\omega = \omega e^{-2u} - \omega \cosh(2u) \\ &= -\omega \sinh(2u) = -\bar{\partial}(ie^{-u}\psi_2\varphi_1 + i\lambda^{-1}e^u\psi_1\varphi_2),\end{aligned}$$

we see that up to a complex constant $-2Q\tau + \partial\omega = ie^{-u}\psi_2\varphi_1 + i\lambda^{-1}e^u\psi_1\varphi_2$, and consequently obtain the formula for τ . A similar computation yields the formula for σ . To prove (iii) we compute

$$\frac{1}{2}\partial\tau + \tau\partial u = -\frac{\lambda^{-1}}{4Q}\omega, \quad \frac{1}{2}\bar{\partial}\sigma + \sigma\bar{\partial}u = -\frac{\lambda}{4Q}\omega.$$

From (2.6) we have $\dot{u} = \frac{1}{2}\partial\tau + \tau\partial u + \frac{1}{2}\bar{\partial}\sigma + \sigma\bar{\partial}u - \omega H$. A straightforward computation using (1.8) and (1.10) now proves (3.4).

From equations (3.1) and (3.2) and (3.3) we obtain

$$\begin{aligned}\partial\omega &= ie^{-u}\varphi_1\psi_2 - i\lambda^{-1}e^u\varphi_2\psi_1 = Q\tau + \lambda^{-1}e^{2u}\bar{Q}\sigma, \\ \phi &= \partial\omega - 2Q\tau = -Q\tau + \lambda^{-1}e^{2u}\bar{Q}\sigma.\end{aligned}\tag{3.6}$$

Consequently $\tau\sigma$ is equal to

$$\tau\sigma = \frac{1}{2Q}(\partial\omega - \phi)\frac{\lambda}{2e^{2u}\bar{Q}}(\partial\omega + \phi) = \lambda v^{-2}((\partial\omega)^2 - \phi^2).$$

On the other hand equations (3.3) yield

$$\tau\sigma = \frac{e^{-2u}}{Q\bar{Q}}\psi_1\psi_2\varphi_1\varphi_2 = \frac{e^{-2u}}{4Q\bar{Q}}(y^2 - \omega^2) = v^{-2}(y^2 - \omega^2),$$

and equating these proves (iv). To prove (v), we compute $\partial\phi = \partial^2\omega - 2Q\partial\tau = \partial^2\omega - \partial^2\omega + 2\partial u\partial\omega = 2\partial u\partial\omega$, and $(-\lambda^{-1})\omega = 2Q\partial\tau + 4Q\tau\partial u = \partial^2\omega - \partial\phi + 2\partial\omega\partial u - 2\phi\partial u = \partial^2\omega - 2\phi\partial u$. \square

3.1. Involutions. In the sequel we shall consider solutions u of (1.6) corresponding to a spectral curve with involutions σ , η and ϱ and not necessarily normalized Baker-Akhiezer function ψ . The maps $\psi = (\psi_1, \psi_2)^t$ as in (3.1) and $\varphi = (\varphi_1, \varphi_2)^t$ as in (3.2) transform as follows under the involutions of the spectral curve:

$$\begin{aligned}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, & \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \\ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \eta^* \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \end{pmatrix}, & \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \eta^* \begin{pmatrix} \overline{\varphi_1} \\ \overline{\varphi_2} \end{pmatrix}, \\ \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varrho^* \begin{pmatrix} \overline{\psi_1} \\ \overline{\psi_2} \end{pmatrix}.\end{aligned}$$

If u is periodic with respect to at least one period γ , then the eigenvalues of the corresponding monodromy define the spectral curve with these involutions σ , η and ϱ . In this case the solutions ψ of (3.1) and φ of (3.2) diagonalize the translations by the periods with inverse eigenvalues μ and μ^{-1} . Hence up to normalization ψ is the Baker-Akhiezer function.

Proposition 3.2. *Let u be a solution of (1.6) of finite type and $\psi = (\psi_1, \psi_2)^t$ respectively $\varphi = (\varphi_1, \varphi_2)^t$ be the corresponding solutions of (3.1) and (3.2). We define*

$$P = \frac{\psi \varphi^t}{\psi^t \varphi}. \quad (3.7)$$

Then

$$\begin{aligned} \partial u &= \frac{i}{2} \operatorname{res}_{\lambda=0} \left(\lambda^{-1} d\lambda^{1/2} \operatorname{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \right) \right), \\ \bar{\partial} u &= -\frac{i}{2} \operatorname{res}_{\lambda=\infty} \left(\lambda d\lambda^{-1/2} \operatorname{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \right) \right). \end{aligned} \quad (3.8)$$

Proof. Define $\tilde{\psi}_1, \tilde{\psi}_2$ and $\tilde{\varphi}_1, \tilde{\varphi}_2$ by the gauge

$$\begin{aligned} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} &= \begin{pmatrix} \lambda^{-1/2} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} &= \begin{pmatrix} \lambda^{1/2} e^{-u/2} & 0 \\ 0 & e^{u/2} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \end{aligned} \quad (3.9)$$

Then $\tilde{\psi}_1, \tilde{\psi}_2$ and $\tilde{\varphi}_1, \tilde{\varphi}_2$ are solutions of

$$\begin{aligned} \partial \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 2\partial u & i\lambda^{-1/2} \\ i\lambda^{-1/2} & -2\partial u \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}, \quad \bar{\partial} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & i\lambda^{1/2} e^{2u} \\ i\lambda^{1/2} e^{-2u} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}, \\ \partial \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} 2\partial u & i\lambda^{-1/2} \\ i\lambda^{-1/2} & -2\partial u \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}, \quad \bar{\partial} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & i\lambda^{1/2} e^{-2u} \\ i\lambda^{1/2} e^{2u} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix}. \end{aligned}$$

Hence the asymptotic expansions at $\lambda = 0$ are

$$\begin{aligned} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} &= \exp \left(\frac{iz}{2} \lambda^{-1/2} \right) \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\lambda^{1/2} \begin{pmatrix} \partial u \\ -\partial u \end{pmatrix} + O(\lambda) \right), \\ \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} &= \exp \left(-\frac{iz}{2} \lambda^{-1/2} \right) \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\lambda^{1/2} \begin{pmatrix} \partial u \\ -\partial u \end{pmatrix} + O(\lambda) \right). \end{aligned}$$

Then

$$\tilde{P} = \frac{\tilde{\psi} \tilde{\varphi}^t}{\tilde{\psi}^t \tilde{\varphi}} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - i\lambda^{1/2} \begin{pmatrix} \partial u & 0 \\ 0 & -\partial u \end{pmatrix} + O(\lambda)$$

and consequently

$$\begin{aligned} P &= \begin{pmatrix} \lambda^{1/2} e^{-u/2} & 0 \\ 0 & e^{u/2} \end{pmatrix} \tilde{P} \begin{pmatrix} \lambda^{-1/2} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & \lambda^{1/2} e^{-u} \\ \lambda^{-1/2} e^u & 1 \end{pmatrix} - i\lambda^{1/2} \begin{pmatrix} \partial u & 0 \\ 0 & -\partial u \end{pmatrix} + O(\lambda), \end{aligned} \quad (3.10)$$

which implies the first equation in (3.8). Analogous computations at $\lambda = \infty$ prove the second equation in (3.8). \square

Recall the iteration in Pinkall–Sterling [9] that generates a sequence of solutions to the homogeneous Jacobi equation: Starting with the 'trivial' solution $\omega_1 = \partial u$, then $\varphi_1 = (\partial u)^2$ and thus with the slightly different normalization in our setting we obtain $\tau_1 = \partial\omega_1 - \varphi_1$, which yields a second solution

$$\omega_2 = \partial^3 u - 2(\partial u)^3$$

of the homogeneous Jacobi equation. This iterative procedure at each step requires for a given ω_n to find τ_n solving both

$$\begin{aligned}\partial\tau_n &= \partial^2\omega_n - 2\partial\omega_n\partial u, \\ \bar{\partial}\tau_n &= -4Q\bar{Q}v^{-2}\omega_n = -e^{-2u}\omega_n,\end{aligned}$$

and then defining $\omega_{n+1} = \partial\tau_n + 2\tau_n\partial u$. To solve these equations for τ_n , it is useful to introduce the auxiliary functions ϕ_n such that

$$\tau_n = \partial\omega_n - \phi_n. \quad (3.11)$$

Then ϕ_n satisfies

$$\begin{aligned}\partial\phi_n &= 2\partial\omega_n\partial u, \\ \bar{\partial}\phi_n &= -\omega_n \sinh(2u).\end{aligned}$$

To supplement ω_n and τ_n at each step to a parametric Jacobi field requires finding a function σ_n satisfying

$$\begin{aligned}\partial\sigma_n &= -4Q\bar{Q}v^{-2}\omega_n, \\ \bar{\partial}\sigma_n &= \bar{\partial}^2\omega_n - 2\bar{\partial}\omega_n\bar{\partial}u.\end{aligned}$$

In analogy to Lemma 3.2 in Pinkall and Sterling [9], we obtain the formula

$$\sigma_n = -e^{-2u}(\partial\omega_{n-1} + \phi_{n-1}). \quad (3.12)$$

We shall now see, that all these solutions fit together in the Taylor expansions of the functions ω , τ , ϕ at $\lambda = 0$:

Let

$$\Phi = \sum_{n=0}^{\infty} (-1)^n \phi_n \lambda^n, \quad \Omega = \sum_{n=0}^{\infty} (-1)^n \omega_n \lambda^n.$$

Proposition 3.3. *Let $y = \psi^t \varphi$. Then the entries of*

$$P = \frac{\psi \varphi^t}{\psi^t \varphi} = \frac{1}{y} \begin{pmatrix} \psi_1 \varphi_1 & \psi_1 \varphi_2 \\ \psi_2 \varphi_1 & \psi_2 \varphi_2 \end{pmatrix}$$

have at $\lambda = 0$ the asymptotic expansions

$$\begin{aligned}
-\frac{i\omega}{2y} &= \frac{1}{\sqrt{\lambda}} \sum_{n=1}^{\infty} \omega_n (-\lambda)^n, \\
-\frac{iQ\tau}{y} &= e^{-u} \psi_2 \varphi_1 = \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \tau_n (-\lambda)^n, \\
-\frac{i\bar{Q}\sigma}{y} &= -e^{-u} \psi_1 \varphi_2 = \frac{1}{\sqrt{\lambda}} \sum_{n=1}^{\infty} \sigma_n (-\lambda)^n, \\
-\frac{i\phi}{2y} &= \frac{iQ\tau - i\lambda^{-1}e^{2u}\bar{Q}\sigma}{2y} = \frac{-e^{-u}\psi_2\varphi_1 - \lambda^{-1}e^u\psi_1\varphi_2}{2y} = \frac{1}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \phi_n (-\lambda)^n.
\end{aligned} \tag{3.13}$$

Proof. Due to (3.6) the coefficients of the series in (3.13) obey the equations (3.11) and (3.12). Hence it suffices to show that these Taylor coefficients obey the recursion formula [9, Proposition 3.1]. The first and last equations in (3.13) read $\omega = 2yi\lambda^{-1/2}\Omega$ and $\phi = 2yi\lambda^{-1/2}\Phi$. From Proposition 3.1 (iv) we have $(\partial\omega)^2 - \phi^2 = \lambda^{-1}(y^2 - \omega^2)$ and thus $\Phi^2 - (\partial\Omega)^2 = \frac{1}{4} + \lambda^{-1}\Omega^2$, or in terms of series

$$\left(\sum_{n=0}^{\infty} (-1)^n \phi_n \lambda^n \right)^2 - \left(\sum_{n=1}^{\infty} (-1)^n \partial\omega_n \lambda^n \right)^2 = \frac{1}{4} + \lambda^{-1} \left(\sum_{n=1}^{\infty} (-1)^n \omega_n \lambda^n \right)^2.$$

Due to (3.10) we conclude $\phi_0 = -\frac{1}{2}$. Consequently, for all $n \in \mathbb{N}_0$ the coefficient ϕ_{n+1} is equal to the coefficient of $(-\lambda)^{n+1}$ of the series

$$\left(\sum_{n=1}^{\infty} (-1)^n \phi_n \lambda^n \right)^2 - \left(\sum_{n=1}^{\infty} (-1)^n \partial\omega_n \lambda^n \right)^2 - \lambda^{-1} \left(\sum_{n=1}^{\infty} (-1)^n \omega_n \lambda^n \right)^2.$$

□

The involution ϱ allows us to compute the asymptotic expansions at $\lambda = \infty$ from the the asymptotic expansions at $\lambda = 0$. Note that $\bar{\tau} = \sigma$ and thus $\varrho^*(\frac{\bar{\omega}}{y}) = \frac{\omega}{y}$ and $\varrho^*(\frac{\bar{\tau}}{y}) = \frac{\sigma}{y}$. We summarize the asymptotic expansions at $\lambda = \infty$ in the following

Corollary 3.4. *At $\lambda = \infty$ we have the asymptotic expansions*

$$\begin{aligned}
\frac{i\omega}{2y} &= \lambda^{1/2} \sum_{n=1}^{\infty} (-1)^n \overline{\omega_n} \lambda^{-n}, \\
\frac{iQ\tau}{y} &= \lambda^{1/2} \sum_{n=1}^{\infty} (-1)^n \overline{\sigma_n} \lambda^{-n}, \\
\frac{i\bar{Q}\sigma}{y} &= \lambda^{1/2} \sum_{n=0}^{\infty} (-1)^n \overline{\tau_n} \lambda^{-n}.
\end{aligned} \tag{3.14}$$

Utilizing the fact that

$$\frac{\omega}{y} = \text{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \right), \tag{3.15}$$

we obtain the following

Corollary 3.5. *We have the asymptotic expansions*

$$\begin{aligned} \operatorname{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \right) &= 2i\lambda^{-1/2} \left(-\lambda \partial u + \lambda^2 (\partial^3 u - 2(\partial u)^3) + O(\lambda^3) \right) \quad \text{at } \lambda = 0, \\ \operatorname{tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P \right) &= -2i\lambda^{1/2} \left(-\lambda^{-1} \bar{\partial} u + \lambda^{-2} (\bar{\partial}^3 u - 2(\bar{\partial} u)^3) + O(\lambda^{-3}) \right) \quad \text{at } \lambda = \infty. \end{aligned}$$

The poles of P are at the branch points of the spectral curve. Clearly the function $y = \psi^t \varphi$ is antisymmetric with respect to the hyperelliptic involution, and has zeroes at the branch points a_1, \dots, a_{2g} . For doubly periodic solutions u of (1.6) the differentials $d \ln \mu_k$, $k = 1, 2$ can be locally written as

$$d \ln \mu_k = \beta_k(y) dy,$$

and if γ_1 and γ_2 denote the periods, there exist two differentials dp^\pm which are holomorphic at $\lambda = \infty$ respectively $\lambda = 0$ and

$$d \ln \mu_k = \gamma_k dp^+ + \bar{\gamma}_k dp^-. \quad (3.16)$$

If Γ is the lattice generated by γ_1 and γ_2 , $\mathbb{T} = \mathbb{C}/\Gamma$, and $f : \mathbb{T} \rightarrow \mathbb{C}$, then we denote

$$\langle f \rangle = \frac{1}{\operatorname{area}(\mathbb{T})} \int_{\mathbb{T}} f dS.$$

For a period γ let Δ_γ denote the period defect, which is the difference between the identity operator and the translation by γ .

Proposition 3.6. *The differentials dp^\pm have the following asymptotic expansions*

$$\begin{aligned} dp^+ &= d\lambda^{1/2} \left(-\frac{i}{2} \lambda^{-1} - i \langle (\partial u)^2 \rangle + O(\lambda) \right) & \text{at } \lambda = 0, \\ dp^- &= d\lambda^{1/2} \left(\frac{i}{2} \langle \cosh(2u) \rangle + O(\lambda) \right) & \text{at } \lambda = 0, \\ dp^+ &= d\lambda^{-1/2} \left(\frac{i}{2} \langle \cosh(2u) \rangle + O(\lambda^{-1}) \right) & \text{at } \lambda = \infty, \\ dp^- &= d\lambda^{-1/2} \left(-\frac{i}{2} \lambda - i \langle (\partial u)^2 \rangle + O(\lambda^{-1}) \right) & \text{at } \lambda = \infty. \end{aligned}$$

Proof. Let $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2)$ as in (3.9) and define $\hat{\psi}$ by $\tilde{\psi} = \exp(p^+ z + p^- \bar{z}) \hat{\psi}$. Since μ_k is the automorphy factor of $\tilde{\psi}$ with respect to γ_k , then $\Delta_{\gamma_k} \hat{\psi} = 0$, and we expand

$$\begin{aligned} \hat{\psi}(z, \lambda) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\sum a_n(z) \lambda^{n/2} \right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(\sum b_n(z) \lambda^{n/2} \right) \quad \text{with } a_0 = 1, b_0 = 0, \\ p^+(\lambda) &= \sum p_n^+ \lambda^{n/2}, \quad p^-(\lambda) = \sum p_n^- \lambda^{n/2} \quad \text{and } p_{-1}^+ = \frac{i}{2}, p_{-1}^- = 0. \end{aligned}$$

The differential equations for $\tilde{\psi}$ and comparison of like coefficients yields

$$\begin{aligned} \partial a_n &= b_n \partial u - p_1^+ a_{n-1} - \dots - p_n^+ a_0, \\ i b_{n+1} &= a_n \partial u - p_1^+ b_{n-1} - \dots - p_n^+ b_1 - \partial b_n, \\ \bar{\partial} a_n &= -p_1^- a_{n-1} - \dots - p_n^- a_0 + \frac{i}{2} (a_{n-1} \cosh(2u) - b_{n-1} \sinh(2u)), \\ \bar{\partial} b_n &= -p_1^- b_{n-1} - p_3^- b_{n-3} - \dots - p_{n-1}^- b_1 + \frac{i}{2} (a_{n-1} \sinh(2u) - b_{n-1} \cosh(2u)). \end{aligned}$$

In particular, $b_1 = -i \partial u$. Further $\partial a_1 = -i(\partial u)^2 - p_1^+$, so integration yields $p_1^+ = -i \langle (\partial u)^2 \rangle$. Thus $\partial a_1 = -i(\partial u)^2 - i \langle (\partial u)^2 \rangle$ which gives

$$\bar{\partial} \partial a_1 = i \partial u \sinh(2u) = \partial \frac{i}{2} \cosh(2u)$$

and therefore $\bar{\partial}a_1 = -p_1^- + \frac{i}{2} \cosh(2u)$ and consequently $p_1^- = \langle \frac{i}{2} \cosh(2u) \rangle$. This proves the asymptotic expansions at $\lambda = 0$. The asymptotic expansions at $\lambda = \infty$ follow from the fact that

$$\varrho^* \overline{d \ln \mu} = -d \ln \mu,$$

and therefore $\varrho^* \overline{dp^+} = -dp^-$ and $\varrho^* \overline{dp^-} = -dp^+$. \square

We now compute the variation of the Baker-Akhiezer function ψ that corresponds to the solution $\dot{u} = (\psi_1 \varphi_1 - \psi_2 \varphi_2)|_{\lambda=a}$ of the homogeneous Jacobi equation at some fixed value $\lambda = a \in \mathbb{C}^*$.

Proposition 3.7. *Let u be a solution of the sinh-Gordon equation (1.6). Let $\xi_{ij} = (\psi_i \varphi_j)|_a$ and $\dot{u} = \xi_{11} - \xi_{22}$ be given in terms of solutions $\psi = (\psi_1, \psi_2)^t$ of (3.1) and $\varphi = (\varphi_1, \varphi_2)^t$ of (3.2) for a fixed value $\lambda = a \in \mathbb{C}^*$. Then the variation of the corresponding solution ψ of (3.1) is given by*

$$\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} = \frac{1}{\lambda - a} \begin{pmatrix} (\lambda + a) \xi_{11} & 2\lambda \xi_{12} \\ 2a \xi_{21} & (\lambda + a) \xi_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (3.17)$$

Proof. By (3.1) we have that

$$\begin{aligned} \partial \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} \partial u & ie^{-u} \\ i\lambda^{-1}e^u & -\partial u \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \partial \dot{u} & -i\dot{u}e^{-u} \\ i\dot{u}\lambda^{-1}e^u & -\partial \dot{u} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\ \bar{\partial} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} -\bar{\partial} u & i\lambda e^u \\ ie^{-u} & \bar{\partial} u \end{pmatrix} \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\bar{\partial} \dot{u} & i\dot{u}\lambda e^u \\ -i\dot{u}e^{-u} & \bar{\partial} \dot{u} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned}$$

Consequently,

$$\partial \dot{u} = ie^{-u} \xi_{21} - ia^{-1} e^u \xi_{12}, \quad \bar{\partial} \dot{u} = ia e^u \xi_{21} - ie^{-u} \xi_{12}. \quad (3.18)$$

Let $X = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be the matrix such that $\dot{\psi} = X \psi$. Then

$$\begin{aligned} 2\partial \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \partial u & ie^{-u} \\ i\lambda^{-1}e^u & -\partial u \end{pmatrix} \right] &= \begin{pmatrix} \partial \dot{u} & -i\dot{u}e^{-u} \\ i\dot{u}\lambda^{-1}e^u & -\partial \dot{u} \end{pmatrix}, \\ 2\bar{\partial} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} -\bar{\partial} u & i\lambda e^u \\ ie^{-u} & \bar{\partial} u \end{pmatrix} \right] &= \begin{pmatrix} -\bar{\partial} \dot{u} & i\dot{u}\lambda e^u \\ -i\dot{u}e^{-u} & \bar{\partial} \dot{u} \end{pmatrix}. \end{aligned} \quad (3.19)$$

We compute the commutators

$$\begin{aligned} \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \partial u & ie^{-u} \\ i\lambda^{-1}e^u & -\partial u \end{pmatrix} \right] &= \begin{pmatrix} \beta i\lambda^{-1}e^u - \gamma ie^{-u} & (\alpha - \delta)ie^{-u} - 2\beta \partial u \\ 2\gamma \partial u + (\delta - \alpha)i\lambda^{-1}e^u & \gamma ie^{-u} - \beta i\lambda^{-1}e^u \end{pmatrix}, \\ \left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} -\bar{\partial} u & i\lambda e^u \\ ie^{-u} & \bar{\partial} u \end{pmatrix} \right] &= \begin{pmatrix} \beta ie^{-u} - \gamma i\lambda e^u & (\alpha - \delta)i\lambda e^u + 2\beta \bar{\partial} u \\ (\delta - \alpha)ie^{-u} - 2\gamma \bar{\partial} u & \gamma i\lambda e^u - \beta ie^{-u} \end{pmatrix}. \end{aligned}$$

Thus equations (3.19) read

$$\begin{aligned} 2\partial \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} (\gamma + \xi_{21})ie^{-u} - (a^{-1}\xi_{12} + \beta\lambda^{-1})ie^u & (\xi_{22} - \xi_{11} - \alpha + \delta)ie^{-u} + 2\beta \partial u \\ (\xi_{11} - \xi_{22} + \alpha - \delta)i\lambda^{-1}e^u - 2\gamma \partial u & (a^{-1}\xi_{12} + \beta\lambda^{-1})ie^u - (\xi_{21} + \gamma)ie^{-u} \end{pmatrix}, \\ 2\bar{\partial} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} (\xi_{12} - \beta)ie^{-u} + (\gamma\lambda - a\xi_{21})ie^u & (\xi_{11} - \xi_{22} - \alpha + \delta)i\lambda e^u - 2\beta \bar{\partial} u \\ (\alpha - \delta + \xi_{22} - \xi_{11})ie^{-u} + 2\gamma \bar{\partial} u & (\beta - \xi_{12})ie^{-u} + (a\xi_{21} - \gamma\lambda)ie^u \end{pmatrix}. \end{aligned}$$

We make the Ansatz $\alpha = A\xi_{11}$, $\beta = B\xi_{12}$, $\gamma = C\xi_{21}$ and $\delta = A\xi_{22}$, and use the fact that

$$\begin{aligned} 2\partial\xi_{11} &= \xi_{21}ie^{-u} - a^{-1}\xi_{12}ie^u, & 2\bar{\partial}\xi_{11} &= -\xi_{12}ie^{-u} + a\xi_{21}ie^u, \\ 2\partial\xi_{12} &= 2\xi_{12}\partial u + (\xi_{22} - \xi_{11})ie^{-u}, & 2\bar{\partial}\xi_{12} &= -2\xi_{12}\bar{\partial}u + a(\xi_{22} - \xi_{11})ie^u, \\ 2\partial\xi_{21} &= -2\xi_{21}\partial u + a^{-1}(\xi_{11} - \xi_{22})ie^u, & 2\bar{\partial}\xi_{21} &= 2\xi_{21}\bar{\partial}u + (\xi_{11} - \xi_{22})ie^{-u}, \\ 2\partial\xi_{22} &= -\xi_{21}ie^{-u} + a^{-1}\xi_{12}ie^u, & 2\bar{\partial}\xi_{22} &= \xi_{12}ie^{-u} - a^{-1}\xi_{21}ie^u. \end{aligned}$$

Comparison of coefficients of e^{-u} , e^u reduce to the three equations

$$A - C = 1, \quad B\lambda^{-1} = a^{-1}(A - 1) \text{ and } B = A + 1.$$

Solving these yields the claim and concludes the proof. \square

The Fermi curve contains a lot of information of the spectral theory of the Jacobi operator. In particular, it should be possible to determine the Maslov index in terms of the spectral curve of the solution of the sinh-Gordon equation.

Lemma 3.8. *Let u be a doubly periodic solution of the sinh-Gordon equation, and let $\psi = (\psi_1, \psi_2)^t$ be a solution of (3.1). Then $\psi_1\psi_2$ is the Baker-Akhiezer function of the Fermi curve of the Jacobi operator.*

Proof. A straightforward computation shows that if $\psi = (\psi_1, \psi_2)^t$ solves (3.1), then $\psi_1\psi_2$ is in the kernel of the Jacobi operator. With respect to the two periods γ_1, γ_2 of u we have $\psi(z + \gamma_j) = \mu_j \psi(z)$, and thus for $j = 1, 2$ we have

$$(\psi_1\psi_2)(z + \gamma_j) = \mu_j^2 (\psi_1\psi_2)(z). \quad (3.20)$$

Hence the spectral curve is a finite covering of the Fermi curve of the Jacobi operator. By Theorem 17.9 in [7], the kernel of the Jacobi operator is generically one dimensional, and therefore the spectral curve is a simple covering of the Fermi curve. \square

Let us finally indicate, how we may construct inhomogenous Jacobi fields. Denote the real quantity

$$\mathfrak{H} = -\frac{\dot{H}}{2(H^2 + 1)} = \frac{\dot{Q}}{2HQ}.$$

Let $\hat{\omega} = \omega^+ + \omega^-$ where $\omega^+ = \mathfrak{H}(z\partial u - \frac{1}{2})$ and $\omega^- = \mathfrak{H}(\bar{z}\bar{\partial}u - \frac{1}{2})$. A straightforward computation shows that $\hat{\omega}$ is also solution of the inhomogeneous Jacobi equation (2.2). The corresponding $\hat{\tau} = \tau^+ + \tau^-$ and $\hat{\sigma} = \sigma^+ + \sigma^-$ that supplement $\hat{\omega}$ to a parametric Jacobi field are given by $\sigma^+ = \overline{\tau^-}$ and $\sigma^- = \overline{\tau^+}$, where

$$\begin{aligned} \tau^+ &= \frac{\partial\omega^+}{2Q} - \frac{\mathfrak{H}z(\partial u)^2}{2Q} - \frac{\mathfrak{H}}{4Q} \int_0^z 2(\partial u)^2 dw - \cosh(2u) d\bar{w}, \\ \tau^- &= \mathfrak{H}Hz + \frac{\mathfrak{H}}{4Q} \bar{z}e^{-2u}. \end{aligned}$$

The period defects of these functions are given by

$$\begin{aligned}\Delta_\gamma \left(\frac{1}{2} \partial \tau^+ + \tau^+ \partial u \right) &= \frac{\mathfrak{H} \gamma}{4Q} (\partial^3 u - 2(\partial u)^3) - \frac{\mathfrak{H} \partial u}{4Q} \int_z^{z+\gamma} 2(\partial u)^2 dw - \cosh(2u) d\bar{w}, \\ \Delta_\gamma \left(\frac{1}{2} \partial \tau^- + \tau^- \partial u \right) &= \mathfrak{H} H \gamma \partial u, \quad \Delta_\gamma \left(\frac{1}{2} \bar{\partial} \sigma^+ + \sigma^+ \bar{\partial} u \right) = \mathfrak{H} H \bar{\gamma} \bar{\partial} u, \\ \Delta_\gamma \left(\frac{1}{2} \bar{\partial} \sigma^- + \sigma^- \bar{\partial} u \right) &= \frac{\mathfrak{H} \bar{\gamma}}{4\bar{Q}} (\bar{\partial}^3 u - 2(\bar{\partial} u)^3) + \frac{\mathfrak{H} \bar{\partial} u}{4\bar{Q}} \int_z^{z+\gamma} 2(\bar{\partial} u)^2 d\bar{w} - \cosh(2u) dw.\end{aligned}$$

We may add to these inhomogenous Jacobi fields homogenous Jacobi fields with the same period defects and obtain periodic inhomogenous Jacobi fields.

4. POLYNOMIAL KILLING FIELDS

Let $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and denote Laurent polynomials $\mathbb{C}^\times \rightarrow \mathfrak{sl}(2, \mathbb{C})$ with a simple pole at $\lambda = 0$ and normalized leading coefficients, by

$$\Lambda_g = \left\{ \xi_0 : \mathbb{C}^\times \rightarrow \mathfrak{sl}(2, \mathbb{C}) \mid \xi_0(\lambda) = \sum_{j=-1}^g c_j \lambda^j, \lambda^{g-1} \overline{\xi(1/\bar{\lambda})}^t = -\xi(\lambda) \text{ and } c_{-1} = v \right\}.$$

The condition $\lambda^{g-1} \overline{\xi(1/\bar{\lambda})}^t = -\xi(\lambda)$ is a reality condition which ensures that the shifted polynomial $\lambda^{-l} \xi$ takes values in $\mathfrak{su}(2)$ on \mathbb{S}^1 , where

$$l = \begin{cases} \frac{1}{2}(g+1) & \text{if } g \text{ is odd,} \\ \frac{1}{2}(g-1) & \text{if } g \text{ is even.} \end{cases}$$

We denote these skew-hermitian loops by

$$\Lambda_g \mathfrak{su}(2) = \{ \lambda^{-l} \xi_0 \mid \xi_0 \in \Lambda_g \}.$$

By the Symes method [11], the extended framing $F : \mathbb{R}^2 \times \mathbb{C}^\times \rightarrow \mathrm{SU}(2)$ of any harmonic map $\mathbb{R}^2 \rightarrow \mathbb{S}^2$ of finite type (see Burstall and Pedit [5, 4]) is given by the unitary factor of the Iwasawa decomposition of

$$\exp(z \xi_0) = F B \tag{4.1}$$

for some $\xi_0 \in \Lambda_g$. The loop $\xi_0 \in \Lambda_g$ is called a *potential* of the corresponding harmonic map in the generalized Weierstrass representation of Dorfmeister, Pedit and Wu [6]. By a result of Pinkall and Sterling [9], and independently Hitchin [8], all Gauss maps of CMC tori are of finite type, so equivalently one may solve first solve the sinh-Gordon equation to obtain some function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ and then solve $dF = F \alpha$ with $F(0) = \mathbb{1}$ with α as in (1.5) to obtain an extended framing. Recall that a *polynomial Killing field* in this case is a map $\xi : \mathbb{R}^2 \rightarrow \Lambda_g \mathfrak{su}(2)$ which solves

$$d\xi = [\xi, \alpha], \quad \xi(0) = \lambda^{-l} \xi_0 \tag{4.2}$$

with $\xi_0 \in \Lambda_g$. The solution to (4.2) via the Iwasawa decomposition (4.1) is then $\xi = F^{-1} \lambda^{-l} \xi_0 F = B \lambda^{-l} \xi_0 B^{-1}$.

Proposition 4.1. *Let ξ be a polynomial Killing field and ψ_0 the eigenvector and φ_0 the transposed eigenvector of $\xi(0)$:*

$$\xi(0) \psi = \nu \psi, \quad \varphi^t \xi(0) = \nu \varphi^t \quad \text{with} \quad \nu^2 = -\det(\xi(0)) = \det(\xi).$$

Then the eigenvector ψ and transposed eigenvector φ^t of ξ are the solutions of

$$d\psi = -\alpha \psi \text{ with } \psi(0) = \psi_0; \quad d\varphi^t = \varphi^t \alpha \text{ with } \varphi^t(0) = \varphi_0^t.$$

Proof. The unique solutions are given by $\varphi^t = \varphi_0^t F$ and $\psi = F^{-1} \psi_0$, and are eigenvectors of the polynomial Killing field. \square

Conversely, assume we are given α as in (1.5) with some periodic finite gap solution of the sinh-Gordon equation. Then ψ, φ^t are the corresponding Baker-Akhiezer functions if

$$\begin{aligned} d\varphi^t &= \varphi^t \alpha \text{ and } \varphi^t M_F = \nu \varphi^t, \\ d\psi &= -\alpha \psi \text{ and } M_F \psi = \nu \psi, \end{aligned} \tag{4.3}$$

where M_F is the monodromy of the extended framing with respect to the period. Note that ψ and φ^t extend meromorphically into points where M_F is not semi-simple. Furthermore, there exists a unique polynomial Killing field ξ such that

$$\varphi^t \xi = \nu \varphi^t \text{ and } \xi \psi = \mu \psi. \tag{4.4}$$

We call ξ a Killing field for ψ and φ . If $\varphi^t(0) = \varphi_0^t$ and $\psi(0) = \psi_0$, then $Q = \psi \varphi^t$ solves $dQ = [Q, \alpha]$ with $Q(0) = \psi_0 \varphi_0^t$. Note that Q has rank ≤ 1 everywhere. At the zeroes of $\det \xi$ (the branch points of the spectral curve), we shall describe the dynamics of the spectral curve under the isoperiodic deformations in terms of the Baker-Akhiezer functions ψ, φ^t .

Let u be a solution of the sinh-Gordon equation, and $\psi = (\psi_1, \psi_2)^t$ and $\varphi = (\varphi_1, \varphi_2)^t$ be solutions of (3.1) and (3.2) respectively. Let $\xi_{ij} = (\psi_i \varphi_j)|_a$ and $\dot{u} = \xi_{11} - \xi_{22}$. Recall from Proposition 3.7 that

$$\dot{\psi} = \frac{1}{\lambda - a} \begin{pmatrix} (\lambda + a) \xi_{11} & 2\lambda \xi_{12} \\ 2a \xi_{21} & (\lambda + a) \xi_{22} \end{pmatrix} \psi. \tag{4.5}$$

Let y be an arbitrary point on the spectral curve, $a = \lambda(y)$ and $Q = \psi(y) \varphi(y)^t$. Then an easy computation shows that the previous equation (4.5) can be rewritten as

$$\dot{\psi} = \frac{1}{\lambda - a} \left(\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix} Q + Q \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} \right) \psi. \tag{4.6}$$

Lemma 4.2. *The transformation $\dot{\psi}$ is an infinitesimal isospectral transformation, that is, $\dot{u} = (\psi_1 \varphi_1 - \psi_2 \varphi_2)|_a$ is isospectral. All isospectral transformations are obtained by taking linear combinations of such transformations at all branch points a . The isospectral transformations are a g -dimensional space, where g is the arithmetic genus of the spectral curve.*

Proof. Let ξ be a polynomial Killing field for ψ and φ , and $a \in \mathbb{C}^*$ any branch point of the spectral curve. Then $\xi(a) Q = \nu Q = Q \xi(a)$.

Since Q , and the matrices $\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix}$ commute with ξ at $\lambda = a$, the commutator $\left[\left(\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix} Q + Q \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} \right), \xi \right]$ has a zero at $\lambda = a$.

From (4.6) and $\dot{\xi} \psi + \xi \dot{\psi} = \nu \dot{\psi} + \dot{\nu} \psi = \nu \dot{\psi}$, we conclude that

$$\dot{\xi} = \frac{1}{\lambda - a} \left[\left(\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix} Q + Q \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} \right), \xi \right]$$

is a tangent vector in the complexified space of Killing fields at the point ξ with $\dot{\nu} = 0$.

Now we claim that all such transformations are linear combinations of such transformations at all branch points of the spectral curve. The meromorphic map $P = \frac{\psi \varphi^t}{\psi^t \varphi}$ has poles only at the branch points. For $g = 0$, the space of Killing fields is zero dimensional, and there are no infinitesimal isospectral transformations. For $g \geq 1$ there exists for each point a meromorphic differential ω with a first order pole at that point, and first order zeroes at $\lambda = 0$ and $\lambda = \infty$. The sum over all residues of ωP vanishes, and consequently, the value of P at such a point is a linear combination of the values of P at the branch points. This proves the claim.

Finally we remark, that for all holomorphic one-forms ω , the sum over all residues of ωP vanishes. Therefore all non-trivial holomorphic one-forms yield a non-trivial relation on the corresponding infinitesimal transformations. Therefore the space of these transformations span a g -dimensional space of isospectral transformations. \square

In the next lemma we exhibit the generators for which $\dot{u} = \partial(\psi_1 \psi_2)|_a$ is non-isospectral.

Lemma 4.3. *Let u be a solution of the sinh-Gordon equation, and $\psi = (\psi_1, \psi_2)^t$ and $\varphi = (\varphi_1, \varphi_2)^t$ be solutions of (3.1) and (3.2) respectively. Let*

$$Q = \frac{d}{dy} (\psi \sigma^* \varphi^t)|_{\lambda=a} \quad (4.7)$$

be the derivative with respect to a local parameter y at some branch point $\lambda = a \in \Sigma_g$. Then

$$\dot{\psi} = \frac{1}{\lambda-a} \left(\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix} Q + Q \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} \right) \psi \quad (4.8)$$

is an infinitesimal non-isospectral transformation that only moves the branch point a while keeping all the other branch points fixed. The space of all these deformations generates the space of all non-isospectral transformations.

Proof. Since the branch points are zeroes of $d\lambda$, the derivative of λ with respect to y vanishes. Hence we may apply Proposition 3.7 in this situation. Since the square of the eigenvalue ν of ξ is equal to λ^{-1} times a polynomial with respect to λ , whose roots are the branch points in \mathbb{C}^* , we have to show that $2\nu\dot{\nu}$ vanishes at all branch points in \mathbb{C}^* with the exception of $\lambda = a$. Hence $\dot{\nu}$ has to be proportional to $\det(\xi)/(\nu(\lambda - a))$. From (4.6) and $\dot{\xi}\psi + \xi\dot{\psi} = \nu\dot{\psi} + \dot{\nu}\psi = \nu\dot{\psi}$ and $\det(\xi)\xi^{-1} = -xi$, we conclude that

$$\left[\left(\begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix} Q + Q \begin{pmatrix} a & 0 \\ 0 & \lambda \end{pmatrix} \right), \xi \right]$$

has to be proportional to ξ at $\lambda = a$. If we differentiate the equations

$$\xi\psi = \nu\psi \quad \sigma^* \varphi^t \xi = \nu \sigma^* \varphi^t$$

at the branch points, we obtain that the commutator $[Q, \xi]$ is at $\lambda = a$ proportional to ξ . This implies the claim. \square

With the help of the inhomogenous Jacobi fields described at the end of the last section, one may construct non-isospectral deformations of CMC tori in \mathbb{S}^3 .

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